Classification of Surfaces, Math 124, Fall 2009

Hirotaka Tamanoi

1. Plane models

Definition 1.1. A *plane model* P is a polygon whose edges are oriented and labeled. A *word* representing a plane model is a list of edge labels of the plane model read clockwise (or counterclockwise) starting from some vertex, and including an exponent of -1 on the edge label oriented in the opposite direction.

Let X be a space obtained by identifying edges of P carrying the same label in the direction indicated. By simply glancing at the word of a plane mode, we can tell whether X is a surface or not, and if so, whether X is orientable or not. Recall that a surface is a space such that every point has an open disc neighborhood.

Definition 1.2. A surface is non-orientable if it contains a Möbius band in it. Equivalently, a surface is non-orientable if it contains an orientation-reversing curve in it. A surface is orientable if it does not contain Möbius band.

Proposition 1.3. *Let M be a word for a plane model P. Let X be the identification space obtained from P. Then X is a surface if and only if each label appears exactly twice in the word M.*

Suppose X is a surface. X is an orientable surface if and only if every label appears twice in opposite direction. (X *is a nonorientable surface if and only if for some label, two edges sharing that label occur in the same direction.*)

Proof. Suppose on the plane model there is a label, say a, which appear in the same direction. Consider a curve connecting the middle points of these two edges. This becomes an orientation reversing curve on the surface X. So it is non-orientable. (You can also observe that when you thicken the curve, we get a Möbius band on the surface.)

Next, consider an arbitrary curve on a surface X obtained from a plane model in which every label appears twice in opposite direction. If this curve corresponds to a curve in the interior of the plane model, then this curve is always orientation preserving. Suppose this curve crosses an edge labeled a. Since two edges carrying the label a are in opposite direction, right-handed arrows along the curve before crossing the a-edge are still right-handed arrows after crossing the a-edge. Thus, right-handed orientation is always preserved after crossing edges, this curve is an orientation preserving curve. Hence this surface is orientable. □

2. Connected Sum

Let S_1 and S_2 be two surfaces. Remove an open disc (a disc without boundary points) from each of these surfaces, and identify the boundaries. The resulting surface is the connected sum of S_1 and S_2, and denoted by $S_1\#S_2$.

Let T be the torus, P be the projective plane, and K be the Klein bottle.

When two surfaces S and S' are the same surface (homeomorphic), then we write $S \cong S'$.

Proposition 2.1. *The following properties hold.*

(1) $P\#P \cong K$.
(2) $T\#P = K\#P = 3P$

The second relation above $T\#P \cong K\#P$, which are surfaces in 4-dimensional space $\mathbb{R}^4$, is a consequence of $T\#M \cong K\#M$, which are surfaces in $\mathbb{R}^3$ and we can visualize them, where M is a Möbius band. Here $T\#M$ is obtained by attaching a handle to a Möbius band from the same side, and $K\#M$ is obtained by attaching a handle to Möbius band from opposite sides. Since Möbius band is one sided, sliding the one end of the handle in $K\#M$ along M bring it to $T\#M$. We can prove the above two identities using circulation rules discussed below.

Proposition 2.2. *Let M_1 be a word representing a surface X_1, and let M_2 be a word representing a surface X_2. Then the concatenation M_1M_2 represents the connected sum $X_1\#X_2$.*

Example 2.3. (1) The orientable genus g surface Σ_g is a g-fold connected sum of the torus T, $\Sigma_g = T\#T\#\cdots\#T$ where T appears g times. Since a word $aba^{-1}b^{-1}$ represents the torus, the surface Σ_g is represented by a word

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}. \tag{2.1}$$

(2) Let $N_h = P\#P\#\cdots\#P$ be a nonorientable surface of genus h, where the projective plane P appears h times on the right hand side. Since P is represented by a word aa, N_h is represented by a word

$$a_1a_1a_2a_2\cdots a_ha_h. \tag{2.2}$$

3. Classification of Surfaces

A surface is *compact* if it is contained in a ball of finite radius and is a closed subset of $\mathbb{R}^3$.

Theorem 3.1. *Let X be a compact connected surface. If X is orientable, then X is a genus g oriented surface Σ_g for some $g \geq 0$. If X is nonorientable, then X is a genus h nonorientable surface N_h.*

It is a nontrivial fact that every compact connected surface can be represented by a plane model. Assuming this, we show that every plane model gives rise to either Σ_g or N_h.

Let M_1 and M_2 be two words representing surfaces. If the corresponding surfaces are homeomorphic (same), then we write $M_1 \sim M_2$, and say M_1 and M_2 are equivalent. We consider various rules to transform given words. These rules allow us to transform any given word for a surface into the form (2.1) or (2.2), proving the classification theorem for surfaces.

4. Circulation Rules

Let M be a word of a surface. Let capital letters $A, B, C, \ldots$ denote blocks of words, and a lower case letter x denote a single label.

(I) Cycle Rule If $M = AB$, then $M \sim BA$.

(II) Flip Rule $M \sim M^{-1}$.

(III) Sphere Rule $Axx^{-1}B \sim AB$.

Lemma 4.1. *The cycle rule and flip rule can be applied to a block within a word if the block represents a word of a surface. Namely, if $M = ABC$ and B is a block representing a word of a surface. If $B \sim D$, then $ABC \sim ADC$.*

(IV) Cylinder Rule Let $M = AxBCx^{-1}D$. Then $M \sim AxCBx^{-1}D$.

(V) Möbius Rule Let $M = AxBxC$. Then $M \sim AxxB^{-1}C$.

Using these rules, we can quickly verify that $P\#P \cong K$, $T\#P \cong 3P$, as follows. For $K \cong P\#P$, $abab^{-1} \sim aab^{-1}b^{-1}$ by Möbius rule, and this word is recognized as the word for $P\#P$. For $T\#P$, $(aba^{-1}b^{-1})(cc) \sim (cycle)ba^{-1}(b^{-1}cc)a \sim$ (cylinder)$ba^{-1}(cb^{-1}c)a \sim$ (Möbius)$ba^{-1}ccba \sim$ (Möbius)$bbc^{-1}c^{-1}aa$, which is a word for $3P$.