

# Solution to Quiz 1, Math 124, Fall 2009

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[1] The  $M$  be a surface whose plane model is given by a word

$$abcdf^{-1}d^{-1}fg^{-1}cgee^{-1}b^{-1}a^{-1}.$$

- (1) Is  $M$  orientable?
- (2) Use circulation rules to transform this word into a standard form.

**Solution.**

- (1) The word  $c$  repeats twice. Hence the surface is nonorientable.
- (2) First, use the cycle rule to move  $ab$  at the front to the back. Then use sphere rules to cancel  $ee^{-1}$ ,  $a^{-1}a$  and  $b^{-1}b$ . Apply Möbius rule to the letter  $c$ , then to letter  $g$  to get  $ccggdf^{-1}d^{-1}f$ . Note that this word represents  $P\#P\#T$ . To go further, we apply the cycle rule to the surface block after  $cc$  to get  $ccfggdff^{-1}d^{-1}$ . Now apply the cylinder rule to  $f \cdots f^{-1}$  to get  $ccfggdff^{-1}d^{-1}$ . Now apply Möbius rule to  $g$  to get  $ccfggd^{-1}f^{-1}d^{-1}$ . Now apply the cylinder rule to the surface block after  $cc$  to move  $f$  to the end:  $ccfggd^{-1}f^{-1}d^{-1}f$ . Then Möbius rule applied to  $d^{-1}$  gives  $ccfggd^{-1}d^{-1}ff$ , which is the word for  $4P$ .

Note that we can compute the Euler characteristic directly from the word. Edge identification induces vertex identification, and we see that there are 4 vertices after identification. That is, the above plane model induced four vertices in the resulting cell decomposition of the surface. There are 7 edges (1-cells) and one face (2-cell). So the Euler characteristic is  $\chi = \nu - e + f = 4 - 7 + 1 = -2$ .

[2] Give precise definitions of the following concepts. Statements equivalent to definitions are acceptable.

- (1) A continuous map from a topological space  $X$  to a topological space  $Y$ .
- (2) A compact topological space.
- (3) A Hausdorff topological space.
- (4) A connected topological space.
- (5) A limit point of a subset  $A$  of a topological space  $X$ .

**Solution: See textbook for definitions.**

[3] Answer the following questions by yes or no. **Answers are : (1) to (5) and (13), No. The rest, Yes.**

- (1) A torus with a finitely many distinct points removed is compact.  
**No. It is not compact. It is bounded, but not a closed subset of  $\mathbb{R}^3$ .**
- (2) Closed subsets in a Hausdorff space are compact.  
**No.  $\mathbb{R}^n$  is Hausdorff, and a closed subset may not be bounded.**
- (3) Compact subsets of a topological space are closed.  
**No. In Hausdorff spaces, compact subsets are closed. But not in general cases. For example, in finite complement topology on  $\mathbb{R}$ , every subset is compact, but not necessarily closed. See the last problem [6] in this quiz.**
- (4) Let  $Y \subset \mathbb{R}^2$  be the graph of  $y = x^2$  for  $x \geq 0$ . That is  $Y = \{(x, x^2) \mid x \geq 0\}$ . Then  $Y$  is a compact subset of  $\mathbb{R}^2$ .  
**No. It is not bounded.**
- (5) The set of all rational numbers contained in the closed interval  $[0, 1]$  is compact.  
**No. The set is a bounded set, but not closed in  $\mathbb{R}$ .**
- (6) Topologist's closed sine curve is a subset of  $\mathbb{R}^2$  given by  $\{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$ . This subset is a compact subset of  $\mathbb{R}^2$ .  
**Yes. It is a bounded and closed subset of  $\mathbb{R}^2$ . It is also connected, by the way.**
- (7) Let  $f : X \rightarrow Y$  be a continuous onto map from a compact space  $X$  to a space  $Y$ . Then  $Y$  is necessarily compact.  
**Yes. This is one of the basic theorems discussed in class.**
- (8) Hawaiian earring is a subset of  $\mathbb{R}^2$  given by  $\bigcup_{n \geq 1} \{(x, y) \mid (x - \frac{1}{n})^2 + y^2 = (\frac{1}{n})^2\}$ . This is a union of circle converging to the origin. The Hawaiian earring is a compact topological space.  
**Yes. It is a bounded closed subset of  $\mathbb{R}^2$ .**
- (9) A compact subset of  $\mathbb{R}^n$  is bounded.  
**Yes. This is a basic theorem for compact subsets in  $\mathbb{R}^n$ .**
- (10) Let  $h$  be a homeomorphism from the boundary of a closed 2-dimensional disc  $D^2$  to the boundary of a Möbius strip  $M$ . Identify the boundary points of  $D^2$  and  $M$  by  $h$ . The resulting space is a projective plane  $P^2$ .  
**Yes. This is one way to think of the projective plane. To see this, remove a small disc from the center of the disc model of the projective plane. We get an annulus**

with identification along the outer boundary. Apply cut and paste using a horizontal cut. You get a Möbius band.

- (11) Let  $Y$  be a subspace of a topological space  $X$ . Let  $A$  be a subset of  $Y$ . If  $A$  is open in  $Y$  and  $Y$  is open in  $X$ , then  $A$  is necessarily an open subset of  $X$ .

**Yes. Since  $A$  is open in  $Y$ , it is of the form  $A = Y \cap U$ , where  $U$  is open in  $X$ . Since  $Y$  is also open in  $X$ , the intersection  $T \cap U$  is also open in  $X$ . Hence  $A$  is open in  $X$ .**

- (12) The Klein bottle and the projective plane are compact topological spaces.

**Yes. They are bounded closed subsets of  $\mathbb{R}^4$ .**

- (13) The subset  $A = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$  of  $\mathbb{R}^2$  is an open subset of  $\mathbb{R}^2$ .

**No.  $A$  does not contain any open ball centered at any point of  $A$ .**

- (14) Let  $X$  be a Hausdorff space and let  $A$  be a compact subset and let  $x \notin A$ . Then there exists a disjoint open neighborhoods of  $A$  and  $x$ .

**Yes. See a Theorem discussed in class.**

- (15) Let  $X$  be a Hausdorff space and let  $A, B$  be disjoint compact subsets. Then there exist disjoint open subsets  $U$  and  $V$  of  $X$  such that  $A \subset U$  and  $B \subset V$ .

**Yes. See one of the exercise problems.**

- (16) Suppose a continuous map  $f : X \rightarrow Y$  is 1:1 and onto. If  $f$  is a closed map, then  $f$  is a homeomorphism.

**Yes. When  $f$  is closed, it maps closed sets to closed sets. Since  $f$  is bijective, it maps complements of sets to complements. Hence  $f$  maps open sets to open sets. Hence  $f^{-1}$  is continuous.**

- [4] Let  $X$  be a topological space, and let  $A$  and  $B$  be subsets of  $X$ .

- (1) Show that if  $A \subset B$ , then  $\bar{A} \subset \bar{B}$ .

**Proof:** We use a characterization of the closure as the smallest closed subset containing the given set. If  $A \subset B$  then  $A \subset \bar{B}$ . Since  $\bar{B}$  is a closed subset containing  $A$ , it also contains the closure of  $A$ ,  $\bar{A} \subset \bar{B}$ .

**Alternate proof.** Since  $A \subset B \subset \bar{B}$ , we have  $A \subset \bar{B}$ . Now let  $x \in X$  be a limit point of  $A$ . Then every neighborhood of  $x$  intersects with  $A$  nontrivially at a point different from  $x$ . Since  $A \subset B$ , this same neighborhood

intersects with  $B$  at a point different from  $x$ . Hence  $x$  is also a limit point of  $B$ . Thus,  $\overline{A} \subset \overline{B}$ .

- (2) Show that  $\overline{A \cup B} = \overline{A} \cup \overline{B}$  in  $X$ .

**We prove both inclusion relations. Since  $A, B \subset A \cup B \subset \overline{A \cup B}$ , we have  $\overline{A}, \overline{B} \subset \overline{A \cup B}$ . Hence  $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ . For the other inclusion relation, note that  $A \cup B \subset \overline{A} \cup \overline{B}$  and the right hand side is a closed set. Hence it contains the closure of the left hand,  $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ . This proves both inclusion relations.**

**An alternate proof of the second inclusion relation  $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ . We have  $A \cup B \subset \overline{A} \cup \overline{B}$  by (1). We show that a limit point of  $A \cup B$  is contained in  $\overline{A} \cup \overline{B}$ . Suppose not. Then  $x$  is in the complement of  $\overline{A} \cup \overline{B}$ , and since the complement is open, there exists an open neighborhood  $U$  of  $x$  not intersecting with  $\overline{A}$  nor  $\overline{B}$ . Thus  $U$  does not contain any points from  $A$  or  $B$ . Thus  $x$  cannot be a limit point of  $A \cup B$ . This is a contradiction. Hence  $x$  must be contained in  $\overline{A} \cup \overline{B}$ .**

[5] Prove that a continuous map  $f : X \rightarrow Y$  from a compact space  $X$  to a Hausdorff space  $Y$  is closed, that is, for every closed subset  $V$  of  $X$ , its image  $f(V)$  is a closed subset in  $Y$ .

If  $V$  is a closed subset of a compact space  $X$ , then  $V$  is compact, by a theorem. Its continuous image  $f(V)$  is compact, by another theorem. Since  $f(V)$  is a compact subset in a Hausdorff space  $Y$ , it is closed in  $Y$ . Hence  $f$  is closed.

[6] Let  $X$  be the set of real numbers  $\mathbb{R}$  with finite complement topology.

- (1) Describe open sets of finite complement topology on the set  $\mathbb{R}$ . **They are  $\emptyset, \mathbb{R}$  or those subsets  $U$  of  $\mathbb{R}$  such that  $\mathbb{R} - U$  is a finite set.**
- (2) Is  $X$  a Hausdorff topological space? Explain. **No. Any two non empty open sets have nontrivial intersection.**
- (3) Show that  $X$  is a compact topological space. **Let  $\{U_\lambda\}$  be an open cover of  $X$ . and let  $U_{\lambda_0}$  be a nonempty member of the family. Let  $X - U_{\lambda_0}$  consist of points  $\{x_1, \dots, x_r\}$ . Since we have an open cover, for each  $i$ , there exists an open set  $U_{\lambda_i}$  in the family containing  $x_i$ . Then  $\{U_{\lambda_0}, U_{\lambda_1}, \dots, U_{\lambda_r}\}$  is a finite open cover of  $X$ .**

**This proof shows that any topological space with finite complement topology is always compact.**

- (4) Show that every subset of  $X$  is a compact subset. **Do the same argument as above. Or induce a topology on this**

subset from  $X$ . Then it is a topological space with finite complement topology. Hence by (3), it is compact.