Solutions to Homework Problem Set 5

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4.1.8 a. True. b. True. c. True. d. No. e. True.

4.1.9 a. 7 colors. b. No, since $N(\mathbb{T}) = 7$.

4.1.10 $\chi(5\mathbb{T}) = -8$. The corresponding Heawood number is 11. So 11 colors are needed.

4.1.11 a. Since $N(S^2) = 4$, $N(D^2) = 4$. b. Since $N(\mathbb{P}) = 6$, we have $N(\text{Möbius band}) = 6$. In both of these cases, we note that gluing a disc along the boundary of the surfaces gives S^2 and $\mathbb{P}$.

4.1.14 a. Let f_n be the number of faces with n edges, $n \geq 2$. Since the map is trivalent, we have $3v = 2e$. Hence $2 = \chi(S^2) = v - e + f = f - \frac{e}{3}$. Using $f = \sum f_n$ and $2e = \sum n f_n$, we get $4f_2 + 3f_3 + 2f_4 + f_5 = 12 + f_7, +2f_8 + \cdots$. Hence we must have $f_n > 0$ for some $2 \leq n \leq 5$.

Similarly for b.

4.3.2 The complete bipartite graph $K_{m,n}$ has $m + n$ vertices and mn edges.

4.3.3 (a) We have $v - e + f = \chi(M)$ and $3f = 2e$. Erasing f gives $e = 3(v - \chi(M))$.

(b) In a graph, there can be at most one edge between a pair of vertices. When there are v vertices, there can be at most $\binom{v}{2} = v(v-1)/2$ many edges. From (1), $3v - 3\chi(M) = e \leq v(v-1)/2$, or $v^2 - 7v + 6\chi(M) \geq 0$. Solving this, we get $v \geq (7 + \sqrt{49 - 24\chi(M)})/2$.

(c) When $\chi = 2$, $v \geq 4$ from the above inequality. The corresponding values of e, f are $e = 6$, $f = 4$. This is realized by a tetrahedron.

(d) Since $\chi = 1$, we have $v \geq 6$ from the above inequality. This is realized by a regular cell complex of type $(3, 5)$ on $\mathbb{P}$.

(e) Since $\chi = 0$, we have $v \geq 7$. When $v = 7$, the corresponding values are $e = 21$ and $f = 14$. This is realized by the dual of the figure 4.4 in text.

4.3.10 Let a student denoted by a vertex and handshaking between two students by an edge between the corresponding vertices. Then we have a graph. For a vertex v, let $\deg v$ denote the number of edges, the degree of v. Then, we have the formula $2e = \sum_v \deg v$. If just one person shakes hands with odd number of students, then in the corresponding graph, there is exactly one vertex with odd degree, which is not possible due to the above formula.

(b) A general statement is that there are even number of students who shake hands with odd number of others.

4.4.3 For the Petersen graph, $v = 10$ and $e = 15$, and the inequality $e \leq 3v - 6$ is satisfied. So we use other method. Note that the above inequality is based on the fact that $2e \geq 3f$, since every face of an embedded graph has at least 3 edges. For the Petersen graph, every possible face has at least 5 edges. So if there exists an embedding into S^2, then we must have $2e \geq 5f$. Together with $v - e + f = 2$, we get $v - 3e/5 \geq 2$. But this is not satisfied by the Petersen graph. Hence the Petersen graph is not embeddable into S^2.

4.4.4 If the pair e, v does not satisfy $e \leq 3v - 6$, then the graph is not embeddable into S^2. However, the converse is false, as the Petersen graph is not embeddable into S^2.

(a), (d) Not embeddable. For (b) and (c), draw a polygon and join appropriate vertices.

4.4.5 When $v = 6$, there can be at most $3v - 6 = 12$ edges in any graph embeddable into S^2. It is easy to draw such an embedded graph.

4.4.6 (a) We get $e \leq 3v - 6$.

(b) We show by induction on v, that $e = 3v - 6$ with $v \geq 3$ can be realized as an embedded graph. When $v = 3$, the graph is a triangle. Suppose we have an embedded graph with v vertices and $e = 3v - 6$ edges. Place a vertex in the middle of a triangle and join this vertex to the three vertices of the triangle. We have a new graph with $v + 1$ vertices and $3v - 3$ edges. This completes the induction.

(c) Simply remove some edges from the maximal case with $e = 3v - 6$.

4.4.7 Place n vertices $v_1, \ldots, v_n$ on the horizontal line and place one vertex w_1 above and one vertex w_2 below, and join v-vertices and w-vertices.

4.4.8 In any embedding of complete bipartite graph into a surface, every face must have at least 4 edges. Hence we have $2e \geq 4f$. If the surface is S^2, we must have $v - e + f = 2$. Together, we have $e \leq 2v - 4$. $K_{3,3}$ does not satisfy this, since $v = 6$ and $e = 9$. .

4.4.9 Connection gives us $K_{3,3}$. So it is not possible to have the desired connection. Since removing one edge from $K_{3,3}$ makes it possible to embed it, it is possible to have connection without cross over if one homeowner decides not to have telephone.

4.4.12 The embedding of $K_{4,4}$ was described in class.

4.4.15 Since $v = 8$, $e = 28$, from $v - e + f = 0$, we have $f = 20$. But this does not satisfy $3f \leq 2e$. Hence K_8 cannot be embedded into torus.

An embeddeing of K_7 is given by dualizing figure 4.4 in text.

4.4.16 $K_{4,4}$ has $v = 9$ and $e = 20$. From $v - e + f = 0$, we have $f = 11$. In every embedding of a complete bipartite graph into a surface, every face must have at least 4 edges. So we must have $4f \leq 2e$. But this is not satisfied for $K_{4,5}$.

4.5.2 $3\mathbb{T}$.