Solutions to Assignment 3

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2.3.12 You get a disc. Undoing the cutting corresponds to identifying antipodal points along the boundary of the disc.

2.3.13 Cutting along β gives a disc and a Möbius band as in figure 2.21. Cutting further along α corresponds to cutting the Möbius band along the centerline, and gives a cylinder.

2.3.14 The projective plane is obtained by identifying antipodal points along the boundary circle of a disc, in which upper semi-circle is identified with the lower semi-circle. The end points of these semi-circles are identified into a single point in the projective plane $\mathbb{P}$. In $\mathbb{P}$, either of the semi-circle gives a simple closed curve. Now cutting open $\mathbb{P}$ along this curve gives back a disc, which is connected. Hence Jordan separation theorem does not hold.

2.3.20 Two Möbius bands. We can directly try to see this by using a space model of the Klein bottle.

If we use the classification theorem, we can argue as follows. Split the Klein bottle into two identical pieces, call each piece N. Close up the hole in each surface by gluing a disc along their boundaries to get a surface M. Then, by construction we must have $M\#M = K$. By Classification Theorem of surfaces, we must have $M \cong \mathbb{P}$. Thus, N is obtained by removing a disc from $\mathbb{P}$, so it is a Möbius band.

2.3.21 Simply remove a disc around the self intersection.

2.3.22 Do cut and paste. (a) A Möbius band. (b) A cylinder.

2.3.23 Two Möbius bands.

2.5.3 (a) The curve does not disconnect the surface. (b) The curve disconnects the surface. (c) The curve does not disconnect (it is an orientation reversing curve). (d) The curve disconnects the surface.

2.5.4 (a) For a torus, use the a-loop and the b-loop to cut open the torus. The surface remains connected. This is possible for $n\mathbb{T}$ with $n \geq 1$ and $m\mathbb{P}$ with $m \geq 2$. For the sphere and the projective plane, this is not possible. For the sphere, single simple closed curve disconnects the surface. For $\mathbb{P}$, a single simple closed curve may not disconnect, but two simple closed curves always do.

(b) The maximum number of disjoint nondisconnecting simple closed curves on $m\mathbb{P}$ is m. These curves are given as follows. To construct $m\mathbb{P}$, we remove small m open discs from a sphere and along each boundary, antipodal points are identified. Thus, along each boundary circle, a semi-circle is identified with the other semi-circle of the same boundary. These m semi-circles become simple closed curves on $m\mathbb{P}$ after identification. If we cut open $m\mathbb{P}$ along these m simple closed curves, you get a sphere with m bounday circles.

To see that m is the maximum number, first observe the following. On $m\mathbb{P}$, choose an orientation reversing simple closed curve. If we cut open the surface along this curve, we get a surface with a single boundary circle. If we glue a disc along this boundary circle, we get a surface $(m-1)\mathbb{P}$.

Instead, if we start with an orientation preserving simple closed curve and cutting the surface $m\mathbb{P}$ along this curve leaves the surface connected with two boundary circles, then capping the boundary circles gives the surface $(m-2)\mathbb{P}$. To see this, let the resulting surface be M. Then the original surface is obtained by attaching a handle to M, that is, taking a connected sum with $\mathbb{T}$. So we have $M\#\mathbb{T} \cong m\mathbb{P}$. By Classification Theorem, $M \cong (m-2)\mathbb{P}$.

Thus, there can be at most m disjoint nondisconnecting simple closed curves on $m\mathbb{P}$.

2.6.3

(a) $aba^{-1}b^{-1}cc$.
(b) $abab^{-1}cc$.
(c) $aba^{-1}b^{-1}cdc^{-1}d^{-1}efef^{-1}ghgh^{-1}$.
(d) $aba^{-1}b^{-1}cdc^{-1}d^{-1}efe^{-1}f^{-1}gghh$.

2.6.6

(a) Not a surface, since words b, c appear only once.
(b) Not a surface, a_4 appears only once.
(c) An orientable surface.
(d) Nonorientable surface, since b_1^{-1} appears twice.
(e) Not a surface, since the letter b appears three times, including its inverse.

2.6.7 The sphere, the torus, the projective plane, the Klein bottle can be made out of a square.

2.7.3 At the presence pf $\mathbb{P}$, the connected sum with $\mathbb{T}$ is equivalent to the connected sum with $2\mathbb{P}$. So $k\mathbb{P}\#n\mathbb{T} \cong (2n+k)\mathbb{P}$.

2.7.4 Use circulation rules to transform words.

2.7.5 (a) The surface is nonorientable and $\mathbb{K}\#S^2\#\mathbb{T}\#2\mathbb{P} \cong 6\mathbb{P}$.
(b) Many ways to do this.

2.7.7 For even $m \geq 2$, $m\mathbb{P} \cong \mathbb{K}\#(\frac{m-2}{2})\mathbb{T}$, and for odd m, $m\mathbb{P} \cong \mathbb{P}\#(\frac{m-1}{2})\mathbb{T}$.