

Solutions to Homework Problem Set 2

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1.3.33 For a compact subset X of \mathbb{R}^n , we show that any sequence $\{x_n\}$ in X has a subsequence that converges to a point in X . First we may assume that $\{x_n\}$ has infinitely many distinct values. If not, the sequence $\{x_n\}$ repeats the same point infinitely many times, and this subset of $\{x_n\}$ clearly converges (the subsequence is stationary). Since the sequence $\{x_n\}$ as a set consists of infinitely many distinct points in a compact set X , it has a limit point z . Let $B_r(z)$ be an open ball of radius r around z . Since z is a limit point, for any $r > 0$, $B_r(z)$ contains a point in the sequence $\{x_n\}$ different from z . Let x_{i_1} be such a member of the sequence in $B_1(z)$. We construct a subsequence $\{x_{i_k}\}$ as follows. Suppose x_{i_k} has been chosen. The sequence $\{x_n\}_{n > i_k}$ obtained by deleting the finitely many points in the beginning of the sequence still has z as a limit point. Let $x_{i_{k+1}}$ be a member of this truncated sequence inside of $B_{1/(k+1)}(z)$. Then we have $i_1 < i_2 < \dots < i_k < \dots$ and we have a subsequence $\{x_{i_k}\}$, and since $|x_{i_k} - z| < 1/k$ for all k , this subsequence clearly converges to z .

1.3.34

- Deleting any two points disconnects S^1 , but deleting two points from the figure 8 does not necessarily disconnect it. So they are not homeomorphic.
- The Cantor set is closed in \mathbb{R}^1 , where the set B is not closed.
- The sphere S^2 is compact as a bounded closed subset of \mathbb{R}^3 , but \mathbb{R}^2 is not compact.
- The Cantor set is closed in \mathbb{R}^1 , but the set of rationals is not closed in \mathbb{R}^1 .

Theorem: Two disjoint compact subsets of a Hausdorff space always possess disjoint open neighborhoods.

Let A, B be two disjoint compact subsets in X . To prove this theorem, we use the proof of the following Theorem.

Theorem : A compact subset of a Hausdorff space is closed.

In the proof, we showed that for a compact subset A of a Hausdorff space X , and for any point $x \in X - A$, there exist disjoint open sets U, V of X such that $A \subset U$ and $x \in V$.

We apply this result for the compact set A and points $x \in B$. Thus for each choice of $x \in B$, we have disjoint open sets U_x and V_x such that $A \subset U_x$ and $x \in V_x$. Now the family of open sets $\{V_x\}_{x \in B}$ is an open cover of B by open sets in X . So by compactness, we can choose a finite subcover V_{x_1}, \dots, V_{x_n} . Then let $U = \bigcap_{i=1}^n U_{x_i}$ and $V = \bigcup_{i=1}^n V_{x_i}$. Then U, V are disjoint open neighborhoods of A, B , respectively.

1.3.22 Let $B_+ = \{(x, \sin(1/x)) \mid x > 0\}$, $B_- = \{(x, \sin(1/x)) \mid x < 0\}$, and $C = \{(0, y) \mid -1 \leq y \leq 1\}$. Then the closures of B_{\pm} is $\overline{B_+} = B_+ \cup C$ and $\overline{B_-} = B_- \cup C$. Since the closure of connected subset is connected, these closures are connected. Since they have common points, namely C , their union $A = B_+ \cup C \cup B_-$ is connected. Here we used the following theorem.

Theorem: The union of connected subsets with nontrivial intersection is also connected.

Can you prove this?

To see that A is not path connected, see *Basic Topology* on page 62.

1.3.23

- a. Not connected. Each point of A is an open subset of A with respect to the induced topology in A .
- b. Connected. We can use the following theorem.

Theorem: The product $X \times Y$ of two connected spaces X, Y is connected.

For a proof, see *Basic Topology* Theorem 3.26 on page 59. In the present case, we can see that the annulus B can be written as $B = S^1 \times (1, 2]$.

- c. Not connected. There are four connected components (each quadrangle without boundary).
- d. Connected, since it is a continuous image of a connected set \mathbb{R}_+^1 of positive real numbers.
- e. Connected. (The condition of connectedness is]it vacuously satisfied.

1.3.24 1.1.5: Not connected. To see this, let z be an irrational number and let $U = \mathbb{Q} \cap (z, \infty)$ and $V = \mathbb{Q} \cap (-\infty, z)$. Since $z \notin \mathbb{Q}$, we have $\mathbb{Q} = U \cup V$, and U, V are disjoint nonempty open subsets of \mathbb{Q} with respect to the induced topology on \mathbb{Q} from \mathbb{R}^1 .

1.1.16. Connected. Use the above theorem of union of connected sets with common points.

1.1.17. Connected. Again use the same theorem as above.

Other examples are omitted.