Solutions to Homework Problem Set 1

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1.1.4

a. The limit point set is $\{0\}$. A is not closed, nor open.

b. The limit point set is the closed anuulus $\{(x,y) \mid 1 \le x^2+y^2 \le 2\}$. B is not closed nor open.

c. The limit point set of C is the entire plane $\mathbb{R}^2$. C is open, not closed

d. The set of limit points of D is the union of D and the segment of y-axis $\{(0,y) \mid -1 \le y \le 1\}$. D itself is not open nor closed.

e. The limit set is empty (no points satisfy the condition of limit points, because the definition of limit points of A requires that A is not empty). By the topology axioms, the empty set is always both open and closed, so the closure of the empty set is itself, the empty set.

1.1.5 All points in $\mathbb{R}^2$.

1.1.8 Let $\{U_\alpha\}$ be a collection of open sets in $\mathbb{R}^n$. Let U be their union, and let $x \in U$. Then for some index β, $x \in U_\beta$. so there exists $\epsilon > 0$ such that U_β contains the ϵ-ball centered at x. Then the same ϵ ball is contained in U. Thus, every point in U has an open ball contained in U. Hence U is open.

1.1.9 a. Let U be the intersection of open sets $U_1, \dots, U_k$. Let $x \in U$. Since for each i we have $x \in U_i$ and U_i is open, there exists $\epsilon_i > 0$ such that the ϵ_i-ball $U_{\epsilon_i}(x)$ is contained in U_i. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_k\}$. Note that ϵ is positive. Then the ϵ-ball $U_\epsilon(x)$ is contained in all of U_i. Hence it is contained in the intersection U. Hence U is open.

b. Take $U_n = (-1/n, 1/n)$, an open interval for all $n > 0$. Then their intersection consists of $0 \in \mathbb{R}^1$, a single point, and it is not open in $\mathbb{R}^1$.

1.1.12 Suppose C is closed in A. Then $A - C$ is open in A, hence it is of the form $A - C = A \cap U$ for some open set in X (ambient space). This means that $C = A \cap (X - U)$. If we let $D = X - U$, then $C = A \cap D$ and D is closed in X.

1.1.13 U is not open in $\mathbb{R}^2$, since U cannot contain any 2-dimensional disc of any radius. But it is open in $\mathbb{R}^1$, because $U = \mathbb{R}^1 \cap D^2$, where D^2 is an open disc in $\mathbb{R}^2$. Note that the induced topology on $\mathbb{R}^1$ induced from the standard topology in $\mathbb{R}^2$ is the same as the standard topology in $\mathbb{R}^1$. On the other hand, as a subset of D^2 with topology induced from $\mathbb{R}^2$, $U = D^2 \cap \mathbb{R}^1$ is not open in D^2 since $\mathbb{R}^1$ is not open in $\mathbb{R}^2$.

1.1.14 For example, consider the set of integers $\mathbb{Z}$. The topology on $\mathbb{Z}$ induced from $\mathbb{R}^1$ is such that every point of $\mathbb{Z}$ is open, since for every integer n we have $\{n\} = \mathbb{Z} \cap (n - \frac{1}{2}, n + \frac{1}{2})$, where the open interval is open in $\mathbb{R}^1$. There are many other examples.

1.1.18 The limit point set of $\mathbb{Q}$ is the entire real line $\mathbb{R}^1$. $\mathbb{Q}$ itself is not open nor closed in $\mathbb{R}^1$.

The topologists comb A is a closed subset of $\mathbb{R}^2$, and the set of limit points of A is A itself.

The Hawaiian earring B is a closed subset of $\mathbb{R}^2$, and the limit point set if B itself.

1.2.1 Recall that $f : X \to Y$ is continuous if and only if every open set in Y pulls back to an open set in X. Now note that for any set B in Y, the complement $Y - B$ of B pulls back to the complement $f^{-1}(Y - B) = X - f^{-1}(B)$ of the pull-back $f^{-1}(B)$. For a closed set A in Y, write it as $A = Y - U$ for some open set U in Y. Then its pull-back is given by $f^{-1}(A) = f^{-1}(Y - U) = X - f^{-1}(U)$ and $f^{-1}(U)$ is open by continuity of f. Hence $f^{-1}(A) = X - f^{-1}(U)$ is closed in X. The reverse implication is similar.

1.2.4 Let $f : X \to Y$ and $g : Y \to Z$. The their composition is $g \circ f : X \to Z$. For an open set U in Z, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open since $g^{-1}(U)$ is open by continuity of g and $f^{-1}(g^{-1}(U))$ is open by the continuity of f.

1.2.12 The inverse function $W^{-1} : S^1 \to [0, 1)$ is not continuous, since an open subset $U = [0, \frac{1}{2})$ of $[0, 1)$ pulls back to $(W^{-1})^{-1}(U) = W(U)$, which is the upper half of the circle with $(1, 0)$ included, but $(-1, 0)$ is not included. So it is not an open subset of S^1. Note that an open subset of S^1 is an open arc.

1.2.23 Suppose x is a limit point of A in X. This means that every open neighborhood of x contains a point from A different from x. Now let U be an arbitrary open neighborhood of $f(x)$. Then $f^{-1}(U)$ is open and contains x. Since x is a limit point, $f^{-1}(U)$ contains a point of A different from x. Applying f, this means that U contains a point of $f(A)$ different from $f(x)$. Since this is the case for any open neighborhood of $f(x)$, it follows that $f(x)$ is a limit point of $f(A)$. Reverse implication is similar.

1.2.24 Being a homeomorphism means that open sets in X and open sets in Y are in bijective correspondence. By taking complements, closed sets in X and closed sets in Y are in bijective correspondence. Namely, $f : X \to Y$ is a homeomorphism if and only if $A \subset X$ closed implies $f(A) \subset Y$ is closed and vice versa.

1.3.28 Since a bounded open interval $(-1, 1)$ is homeomorphic to the unbounded $\mathbb{R}^1$, boundedness is not preserved under homeomorphisms. So it is not a topological property.

1.3.29 Since $[0, 1]$ is a bounded and closed subset of $\mathbb{R}^1$, it is compact. But $[0, 1)$ is not closed, so it is not compact.

1.3.30 The closed disc is bounded and closed, so it is compact. But $\mathbb{R}^2$ is not bounded (it is closed), so it is not compact. So they cannot be homeomorphic.

1.3.31 1.1.4.a: A is not a closed subset of $\mathbb{R}^1$. So it is not compact.

b: B is not a closed subset of $\mathbb{R}^2$. So it is not closed.

c: C is an open subset of $\mathbb{R}^2$. So it is not compact.

d: C is not closed. So it is not compact.

e: The empty set of closed (it is the complement of the open set $\mathbb{R}^n$). Since the empty set of a subset of any bounded set, it is also bounded. Hence it is compact.

1.1.15: $\mathbb{Q}$ is not closed in $\mathbb{R}^1$. So it is not compact.

1.1.16: The topologist's comb is bounded and closed in $\mathbb{R}^2$. So it is compact.

1.1.17: The Hawaiian earring is a bounded closed subset of $\mathbb{R}^2$. So it is compact.

1.1.27: The quotient map is continuous, and the continuous image of a compact set is compact.

1.1.29: Again by the same reason above, the continuous image of a compact set is compact.

(1) Let X be a set. The finite complement topology on X is defined by $\mathcal{U} = \{U \subset X \mid X - U$ is a finite set or $X\}$. Obviously $X, \emptyset$ are in $\mathcal{U}$. Let $\{U_\lambda\}$ be a sub collection of $\mathcal{U}$. If all of them are empty sets, then their union is also an empty set. If there are nonempty set U_μ among this collection, then their union contains U_μ and hence the union differs from X by at most finitely many points. Hence the union is in $\mathcal{U}$. For the third axiom, let $U_1, \ldots, U_k$ be elements in $\mathcal{U}$. If any of the elements is an empty set, their intersection is an empty set, which is in $\mathcal{U}$. If all of them are not empty sets, then let $U_i = X - F_i$ for $1 \leq i \leq k$, where F_i is a finite subset of X. Then the intersection is given by $U_1 \cap \cdots \cap U_k = X - \cup F_i$, and the finite union $\cup F_i$ is a finite subset of X. Hence the finite intersection belongs to $\mathcal{U}$. Hence $\mathcal{U}$ defined a topology.

(2) Recall that the closure of A is the union of A and its limit points. Also another characterization of the closure of A is as the smallest closed subset containing A. Now, $A \subset B$ and $B \subset \overline{B}$ implies $A \subset \overline{B}$. Since $\overline{B}$ is a closed subset containing A, it contains the closure of A, namely $\overline{A} \subset \overline{B}$.

(Another proof.) Since $A \subset \overline{B}$ is obvious, we show that limit points of A are also limit points of B. To see this, let x be a limit point of A. Thus, every open neighborhood of x contains a point a from A different from x. Since $A \subset B$, the point a also belongs to B. Thus, every open neighborhood of x contains a point from B different from x. Hence x is a limit point of B.

(3) The sphere is given by $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Let $A(x, y, z)$ be a point on the sphere different from the north pole $N(0, 0, 1)$. Let $B(s, t, 0)$ be the point in $\mathbb{R}^2$ corresponding to the point A via stereographic projection. Let $O(0, 0, 0)$ be the origin and let $C(0, 0, z)$ be the point on z axis. Then the triangle NCA is similar to the triangle NOB. Since the vertical edges have length $1 - z$ and 1, respectively, quantities on the larger triangle is obtained by magnifying the small triangle by $1/(1 - z)$. Hence $s = x/(1 - z)$ and $t = y/(1 - z)$.

For the formula for the inverse function, for a given point $B(s, t, 0)$ in the plane, let $A(x, y, z)$ be the corresponding point on the sphere. Then by a similar reasoning using triangle similarity, $x = (1 - z)s$ and $y = t(1 - z)$. Since $x^2 + y^2 + z^2 = 1$, we can express z in terms of s, t. Using this z written i s, t, we can write down x, y in terms of s, t.