Topological Spaces
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1. Topological spaces

A topology on a set X allows you to talk about closeness of two points in X.

Definition 1.1. Let X be a set. A topology on X is a collection $\mathcal{U}$ of subsets of X such that

(i) $\emptyset, X \in \mathcal{U}$.
(ii) $\mathcal{U}$ is closed under arbitrary union. Namely, let $\{U_\lambda\}_\lambda \subset \mathcal{U}$ be an arbitrary subcollection of $\mathcal{U}$. Then their union $\bigcup_\lambda U_\lambda$ is in $\mathcal{U}$.
(iii) $\mathcal{U}$ is closed under arbitrary finite intersection. Namely, for $U_i \in \mathcal{U}$ for $1 \le i \le n$, their intersection $\bigcap_{i=1}^n U_i$ is in $\mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a topological space.

Each element in $\mathcal{U}$ is called an open set of X with respect to the topology $\mathcal{U}$. The above definition and the terminology *open* set comes from the following basis example of a topology in $\mathbb{R}^n$.

For $x, y \in \mathbb{R}^n$, let $|x-y|$ be their Euclidean distance. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, let the ε-ball around x to be

$$U_\varepsilon(x) = \{y \in \mathbb{R}^n \mid |x-y| < \varepsilon\}.$$

Definition 1.2. (1) A subset $U \subset \mathbb{R}^n$ is called an open set if for every $x \in U$, there exists $\varepsilon = \varepsilon_x > 0$ such that $U_\varepsilon(x) \subset U$.

(2) A subset $V \subset \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n - V$ is an open set.

It is straightforward to check that the family of subsets $\mathcal{U} = \{U \subset \mathbb{R}^n \mid U \text{ is an open subset in } \mathbb{R}^n\}$ satisfies the axioms of topology.

We remark that the third axiom of topology is for finite intersections. The following example illustrate this point. Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$ be an open subset for $n = 1, 2, \dots$. Then their infinite intersection $\bigcap U_n = \{0\}$ is not an open subset of $\mathbb{R}$.

On a given set X, in general one can put different topology ot X.

Definition 1.3. (1) (Finite complement topology) For a set X, consider a family of subsets

$$\mathcal{F} = \{U \subset X \mid X - U \text{ is finite or all of } X.\}$$

It can be easily checked that this family of subsets satisfies the axioms of topology.

(2) (Discrete topology) For a set X , the family of subsets $\{U \subset X \mid U \text{ is an arbitrary subset of } X.\}$ defines a topology on X. In this topology, one point subset $\{x\}$ is an open subset. Indeed, every subset is an open subset in discrete topology.

Thus, in $\mathbb{R}^n$, we can consider at least three different topologies, the standard topology, finite complement toplogy, and discrete topology.

Exercise. Show that the finite complement topology satisfies the topology axioms.

Solution. The finite complement topology on X is defined by $\mathcal{U} = \{U \subset X \mid X - U \text{ is a finite set or } X\}$. Obviously $X, \emptyset$ are in $\mathcal{U}$. This verifies the axiom (I). Let $\{U_\lambda\}$ be a sub collection of $\mathcal{U}$. If all of them are empty sets, then their union is also an empty set. If there are nonempty set U_μ among this collection, then their union contains U_μ and hence the union differs from X by at most finitely many points. Hence the union is in $\mathcal{U}$. This verifies the second axiom. For the third axiom, let $U_1, \ldots, U_k$ be elements in $\mathcal{U}$. If any of the elements is an empty set, their intersection is an empty set, which is in $\mathcal{U}$. If all of them are not empty sets, then let $U_i = X - F_i$ for $1 \leq i \leq k$, where F_i is a finite subset of X. Then the intersection is given by $U_1 \cap \cdots \cap U_k = X - \cup F_i$, and the finite union $\cup F_i$ is a finite subset of X. Hence the finite intersection belongs to $\mathcal{U}$. Hence $\mathcal{U}$ defined a topology.

2. Subspaces

Definition 2.1. Let $(X, \mathcal{U})$ be a topological space. Let $Y \subset X$ be a subset. We can define a topology on Y by the following family of subsets

$$\mathcal{U}_Y = \{Y \cap U \mid U \in \mathcal{U}\}.$$

The topological space $(Y, \mathcal{U}_Y)$ is called the subspace of X.

Check that $\mathcal{U}_Y$ defines a topology on Y.

Proposition 2.2. *Let X be a topological space. Let $Y \subset X$ be a subspace. Then any closed subset W of Y is of the form $W = Y \cap V$, where V is closed in X.*

Example 2.3. (1) For $m < n$, the topology on $\mathbb{R}^m$ induced from the standard topology on $\mathbb{R}^n$ is the standard topology on $\mathbb{R}^m$.

(2) Any orientable surface in $\mathbb{R}^3$ and nonorientable surface in $\mathbb{R}^4$ are regarded as topological spaces with respect to the induced topology from the standard topology in $\mathbb{R}^n$.

(3) The topology on subset $\mathbb{Z} \subset \mathbb{R}$ induced from the standard topology on $\mathbb{R}$ is discrete.

(4) Let $Y = (0, 1] \subset \mathbb{R}$ with induced topology. Then a subset $(\frac{1}{2}, 1] = (\frac{1}{2}, \frac{3}{2}) \cap Y$ is an open subset of Y, and $(0, \frac{1}{2}] = [\frac{1}{2}, \frac{1}{2}] \cap Y$ is a closed subset of Y.

3. Continuous maps

The generalization of the concept of continuity is defined by topology.

Definition 3.1. Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be topological spaces. A map $f : X \to Y$ is continuous with respect to the topologies on X and Y if for every open set U of $(Y, \sqcap_Y)$, its inverse image $f^{-1}(U) \subset X$ is open in X with respect to $\mathcal{U}_X$.

The usual definition of continuity of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is equivalent to the above definition with respect to the standard topology on $\mathbb{R}^n$ and $\mathbb{R}^m$. To see this, we recall the usual definition of the continuity.

Definition 3.2. (1) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$. (In terms of neighborhoods, this means $f(U_\delta(a)) \subset U_\varepsilon(f(a))$.)

(2) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$ is f is continuous at every point of $\mathbb{R}^n$.

Proposition 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous everywhere on $\mathbb{R}^n$ in the usual sense if and only if for every open set U of $\mathbb{R}^m$, $f^{-1}(U)$ is open in $\mathbb{R}^n$.*

Let $f : \mathbb{R} \to \mathbb{R}$. If f is not continuous, then $f^{-1}(U)$ is not necessarily open in $\mathbb{R}$.

4. Limit points, closure, and interior

Let $(X,\mathcal{U})$ be a topological space.

Definition 4.1. Let $A \subset X$ be a subset. A point $x \in X$ is a limit point of A if every open set containing x intersects A in some point other than x itself.

Example 4.2. Let $X = \mathbb{R}$ with standard topology.

(1) Let $A = (1, 2)$. Then the set of all limit points of A is $[1, 2]$.
(2) Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then 0 is the only limit point of a.
(3) Let $A = \mathbb{Q}$. Then all points of $\mathbb{R}$ are limit points of $\mathbb{Q}$.
(4) $A = \mathbb{Z}$ does not have limit points.

Definition 4.3. For a subset $A \subset X$, the closure of A is

$$\overline{A} = A \cup \{\text{limit points of } A\}.$$

A subset $A \subset X$ is called dense in X is $\overline{A} = X$.

Theorem 4.4. (1) *The closure $\overline{A}$ of A is a closed subset of X.*
(2) *$\overline{A}$ is the smallest closed set containing A.*

Thus, a subset $A \subset X$ is closed if and only if $A = \overline{A}$.
By the above Theorem, we could define the closure of A by

$$\overline{A} = \bigcap_{\substack{A \subset V \\ V \text{ is closed}}} V.$$

In the similar spirit, we define the interioe of a subset A to be the union of all open sets contained in A.

$$\mathring{A} = \bigcup_{\substack{U \subset A \\ U \text{ is open}}} U.$$

We have $\mathring{A} \subset A \subset \overline{A}$.

Definition 4.5. For a subset $A \subset X$, the frontier (or the boundary) of A is given by

$$Bd(A) = \overline{A} \cap \overline{(X - A)} = \overline{A} - \mathring{A}.$$

For example, the boundary of $\mathbb{Q} \subset \mathbb{R}$ is the entire $\mathbb{R}$.

5. Homeomorphism

From now on, every set is assumed to have a topology, making it into a topological space.

Definition 5.1. A map $f : X \to Y$ is a homeomorphism if

(1) f is 1:1 and onto (bijective),
(2) f is continuous,
(3) f^{-1} is continuous.

Thus, $f : X \to Y$ is homeomorphism if open sets and closed sets in X, Y are in 1:1 correspondence, and X and Y can be thought of as the "same" topological space.

Example 5.2. (1) $f : (-1,1) \to \mathbb{R}$ given by $f(x) = x/(1-x^2)$ or $f(x) = \tan(\pi x/2)$ are homeomorphisms. Thus, any open interval is homeomorphic to $\mathbb{R}$: $(a,b) \cong \mathbb{R}$, here $\cong$ means homeomorphic. Similarly, an n-dimensional open unit ball $\overset{\circ}{D}{}^n = \{x \in \mathbb{R}^b \mid |x| < 1\}$ is homeomorphic to $\mathbb{R}^n$.

(2) An open interval $(0,1)$ and a closed interval $[0,1]$ are not homeomporhic. Why?

(3) Stereographic projection: $h : S^n - \{\text{one point}\} \xrightarrow{\cong} \mathbb{R}^n$ says that n-dimensional sphere minus one point is homeomorphic to $\diagdown^n$. To write down such a map h, let the point on S^n be the north pole p and consider the unit sphere S^n in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ with $x_{n+1} = 0$. For $x = (x_1, \ldots, x_{n+1}) \in S^n$, let $h : S^n - \{p\} \to \mathbb{R}^n$ be given by $h(x) = \frac{x}{1-x_{n+1}} \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$. We can check that h is a homeomorphism. What is the formula for h^{-1}?

(4) There are bijective continuous maps $f : X \to Y$ which are not homeomorphisms. For example, consider $f : [0,1) \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Although f is bijective and continuous, f^{-1} is not continuous, since the image of an open subset $[0, \frac{1}{2})$ of $[0,1)$ is not an open subset of S^1. Here, S^1 inherits topology from the standard topology in $\mathbb{R}^2$.

(5) A knot is a map $f : S^1 \to \mathbb{R}^3$ which is a homeomorphism onto its inage (with respect to the induced topology on the image). You may not be able to deform one knot to the other without cutting it, any two knots are homeomorphic to each other.

Definition 5.3. A map $f : X \to Y$ is open if f maps every open set in X to an open set in Y. f is closed if f maps every closed set in X to a closed set in Y.

For example, the map in example (4) above is not an open map.