

Connected Spaces

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1. CONNECTED AND PATH-CONNECTED SPACES

Definition 1.1. Let X be a topological space. A path γ in X is a continuous function $\gamma : [0, 1] \rightarrow X$. The space X is path connected if for any pair of points $a, b \in X$, there exists a path joining a and b .

The concept of connectedness is slightly weaker than this. The definition is somewhat technical, but it has very nice properties.

Definition 1.2. A topological space is connected if it cannot be written as a disjoint union of two nonempty open sets.

Theorem 1.3. Suppose X is connected and $f : X \rightarrow Y$ be continuous and onto. Then Y is connected. (The continuous image of a connected space is connected.)

Theorem 1.4. Finite intervals (a, b) , $(a, b]$, $[a, b)$, $[a, b]$, half infinite intervals, and \mathbb{R} are all connected.

Proof. We prove connectedness for \mathbb{R} . Other cases are similar. Suppose $\mathbb{R} = U \cup V$, where U, V are disjoint nonempty open subsets. By taking complements, both U and V are closed in \mathbb{R} . We may assume that there exists $a \in U$ and $b \in V$ such that $a < b$. Let S be the set of points in U less than b , and let s be the supremum of S . Then $s \in \overline{U} = U$. Since $s \in U$, we have $s < b$. Then all numbers above s belong to V . Hence $s \in \overline{V} = V$. Hence $U \cap V$ contains s , a contradiction to our hypothesis that U and V are disjoint. \square

Theorem 1.5. Let $Z \subset X$ be a connected dense subset (that is, $X = \overline{Z}$). Then X is connected. In other words, the closure of a connected set is connected.

Proof. Suppose X is not connected. Then we may write $X = U \cup V$, where U, V are disjoint nonempty open subsets. Then $Z = (U \cap Z) \cup (V \cap Z)$, where both $U \cap Z$ and $V \cap Z$ are open in Z . By connectedness of Z , one of them must be empty. We may assume $Z \cap V$ is empty. Then $Z \subset U$. Since V is open, its complement U is closed. Hence $X = \overline{Z} \subset \overline{U} = U$. Hence $X = U$, contradicting to $V \neq \emptyset$. \square

A connected subset of X are those subsets which are connected with respect to induced topology. Thus, the above theorem says that the closure of a connected subset is connected.

Theorem 1.6. A path-connected topological space is connected.

Use connectedness of $[0, 1]$.

The next theorem says that the union of two connected sets with nontrivial intersection is connected.

Theorem 1.7. Let A, B be connected subsets of $X = A \cup B$ such that $A \cap B \neq \emptyset$. Then X is connected.

Proof. Suppose X is not connected and let $X = U \cup V$, where both U and V are nonempty open sets. Since A, B are connected, each A, B must be contained entirely in U or V . Since $A \cap B = \emptyset$, they must be both contained in either U or V . Thus, $U = X$ or $V = X$. This is a contradiction. Hence X is connected. \square

Definition 1.8. A maximal connected subset is called a (connected) component.

Let $C \subset X$ be a component. Then its closure \overline{C} is also connected. Since C is a maximal connected subset, we must have $C = \overline{C}$. Hence each connected component is a closed subset. By the above theorem, every connected set is contained in a maximal connected subset, or a component. Thus, if two components intersect nontrivially, they must be identical. Thus, every space is a disjoint union of its components. Since each component is a closed subset of X , it follows that every component is also open. Thus, every component is both open and closed. Conversely, every open closed subset of X is a union of components.

For example, when $X = (0, 1) \cup [2, 3]$, then each interval is a component of X .

In general, to show that a subset A of a connected space X coincides with X , we show that A is both open and closed. This is a very useful argument, as in the next example.

Example 1.9. The topologist's closed sine curve is connected, but not path connected. See page 62 of *Basic Topology* for explanation.