Compact Topological Spaces

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1. Compactness

Definition 1.1. Let X be a topological space.

1. An open cover (or covering) of X is a collection of open sets whose union is X.
2. A subcover is a subset of of an open cover which still covers X.
3. A finite subcover is a subcover of an open cover consisting of finitely many elements.

One classical theorem is Heine-Borel Theorem

Theorem 1.2 (Heine-Borel)**.** *Every open cover of a closed interval of* $\mathbb{R}$ *admits a finite subcover.*

See §3.2 for a proof. The above theorem is used to define the concept of compactness.

Definition 1.3. A topological space X is compact if every open cover has a finite subcover.

A topological property is a property of a topological space which is inherited through homeomorphisms. Compactness is a topological property. Namely, if $X \cong Y$, homeomorphic and if X is compact, then Y is also compact. We will encounter many more such properties.

Why compactness is a topological property?

Example 1.4. (1) By Heine-Borel Theorem, any closed interval in $\mathbb{R}$ is a compact space (with respect to induced topology).

(2) $\mathbb{R}^n$ is not compact. Consider an open covering $\{U_n\}_n$ where $U_n = \{x \in \mathbb{R}^n \mid |x| < n\}$ is an open ball of radius n. $\{U_n\}_n$ is an open cover, but it does not have a finite subcover.

(3) Since $(a, b) \cong \mathbb{R}$, open interval (a, b) is not a compact space, with respect to induced topology).

(4) $(0, 1]$ is not compact. Consider an open covering $\{U_n\}_n$ where $U_n = (\frac{1}{n}, 1]$ is an open subset of $(0, 1]$ with respect to the induced topology on $[0, 1]$. This is an open cover, but it does not have a finite subcover.

(5) $X = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ is compact with respect to the induced topology.

(6) We will see that a subset $A \subset \mathbb{R}^n$ is compact if and only if A is a closed bounded set. This characterizes compact subsets in $\mathbb{R}^n$.

Definition 1.5. Let X be a topological space and let $A \subset X$ be a subset. If A is compact with respect to the induced topology, then A is called a compact subset.

We say a family of open sets of X is an open cover of a subset $A \subset X$ if its union contains A. Using this terminology, $A \subset X$ is a compact subset of X if and only if every open cover of A by open sets in X has a finite subcover.

2. Properties of compact spaces

The first theorem says that the continuous image of a compact set is compact.

Theorem 2.1. *Let $f : X \to Y$ be continuous. For any compact subset K of X, its image $f(K)$ is a compact subset of Y.*

In particular, if X is compact and f is onto, then Y is compact.

Theorem 2.2. *Let X be a compact space, and let $A \subset X$ be a closed subset. Then A is compact. That is, a closed subset in a compact space is compact.*

Can we switch "closed" and "compact" in the above statement? Namely, is a compact subset of a closed space (which is always the case) a closed subset? This is the case most of the time. But to be precise, we need one concept.

Definition 2.3. A topological space X is called a Hausdorff topological space if for every pair of points $x, y \in X$, there exist disjoint open neighborhoods of x, y. That is, open sets U_x, U_y containing x, y such that $U_x \cap U_y = \emptyset$.

Example 2.4. (1) $\mathbb{R}^n$ with standard topology is a Hausdorff space.

(2) $\mathbb{R}^n$ with finite complement topology is not Hausdorff. In this topology, nonempty open sets are big and they always overlap.

Lemma 2.5. *Every subspace of a Hausdorff space is Hausdorff.*

Theorem 2.6. *A compact subset of a Hausdorff space is closed.*

For example, subsets $(a, b), (a, b]$ are not compact because they are not closed in $\mathbb{R}$. In the above Theorem, Hausdorff condition is necessary. For example, in the finite complement topology (which is not Hausdorff if there are infinitely many elements in the set), every subset is compact, but not every subset is closed.

Corollary 2.7. *Let X be a compact Hausdorff space, and let $A \subset X$ be a subset of X. Then A is compact if and only if A is closed.*

As a nice application of previous theorems, we have the following useful theorem.

Theorem 2.8. *Let X be a compact topological space and let Y be a Hausdorff topological space. Let $f : X \to Y$ be a bijective continuous map. Then f is necessarily a homeomorphism.*

For example, a bijective continuous map $f : [0, 1) \to S^1$ we encountered earlier was not homeomorphism. This is not surprising since $[0, 1)$ is not a compact topological space.

Corollary 2.9. *Let X be compact and let Y be Hausdorff. Suppose $f : X \to Y$ be injective and continuous. Then f is a homeomorphism onto its image $f : X \cong f(X)$.*

For example, consider an injective continuous map $f : S^1 \to \mathbb{R}^3$. Its image is a know. By the above Corollary, f is a homeomorphism onto its image $f(S^1)$, a knot. Thus, every knot is homeomorphic to a circle.

Theorem 2.10 (Bolzano-Weierstrass)**.** *Let X be compact. Then an infinite subset S of X must have a limit point in X.*

Example 2.11. Let $X = (0, 1]$. X is not compact since an infinite set $\{\frac{1}{n} \mid n \geq 1\}$ is an infinite subset without limit points in X.

3. Characterization of compact subsets in $\mathbb{R}^n$

Theorem 3.1. *A subset $K \subset \mathbb{R}^n$ is compact if and only if K is bounded and closed.*

Proof. Suppose $K \subset \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, as a compact subset, K is closed. Cover K by open subsets $U_n = \{x \mid |x| < n\}$ for $n \geq 1$. Since K is compact, it has a finite subcover. In this case, this means that K is contained in some U_n. Thus, K is bounded.

A proof for the other direction requires a fact that a Cartesian product of two compact topological spaces is compact, which we will discuss later. □

As an application, we have Extreme value Theorem.

Theorem 3.2. *Let X be compact and let $f : X \to \mathbb{R}$ be a continuous function. Then f assumes its max and min in X.*

This is becuse $f(X)$ is a compact subset of $\mathbb{R}$, so it is bounded and closed. Thus, it has the maximum value and the minimum value.