# Euler Characteristics of Surfaces and Regular Complexes

## Math 124, Fall 2009

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## 1. **Euler Characteristic of Surfaces**

**Definition 1.1.** Given a cell decomposition of a surface $M$, let $\nu$, $e$, and $f$ be the number of 0-cells, 1-cells, and 2-cells. The Euler characteristic of this cell complex is defined by

$$\chi(M) = \nu - e + f.$$

The Euler characteristic can be defined for any cell complex of any dimension.

**Example 1.2.** Some examples of cell decompositions and their Euler characteristic. Here $S^2$ denotes the sphere, $P$ denotes the projective plane, and $T$ denotes the torus.

(1) $S^2 = e^0 \cup e^2$. So $\chi(S^2) = 2$.
(2) $P = e^0 \cup e^1 \cup e^2$. So $\chi(P) = 1$.
(3) $T = e^0 \cup e^1_a \cup e^1_b \cup e^2$. So $\chi(T) = 0$.

A basic property of the Euler characteristic is that it is a topolological invariant.

**Theorem 1.3.** *The Euler characteristic is independent of the cell complex structure on a topological space $X$, and depends only on the homeomorphism type of $X$.*

Thus in particular, the Euler characteristic of a surface is independent of its cell decomposition.

Since all surfaces are generated by taking connected sum with $T$ or $P$, here is a formula for the Euler characteristic for connected sum of surfaces.

**Theorem 1.4.** *Let $M_1, M_2$ be surfaces. Then*

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2.$$

For example, the above formula implies that for any surface $M$, we have

$$\chi(M\#T) = \chi(M) - 2, \qquad \chi(M\#P) = \chi(M) - 1.$$

Thus, taking connected sum with the torus decreases the Euler characteristic by 2, and taking the connected sum with the projective plane decreases the Euler characteristic by 1. By classification theorem of surfaces, a surface is homeomorphic to either an orientable surface $\Sigma_g = gT$ of genus $g \geq 0$, or a nonorientable surface $N_h = hP$ of genus $h \geq 1$. By convention, $N_0 = \Sigma_0 = S^2$ is the sphere. By applying the above formula for taking connected sum with the torus or with the projective plane, we get the following.

**Proposition 1.5.** *The euler characteristic of surfaces are given as follows.*

$$\chi(\Sigma_g) = 2 - 2g, \qquad \chi(N_h) = 2 - h.$$

Here regard $\Sigma_g$ as $\Sigma_g = S^2\#T\#\cdots\#T$. Same for $N_h$.

## 2. **Applications of Euler Characteristics: Identification of surfaces**

Surfaces can be classified by orientability and the Euler characteristic. So, we can use Euler characteristic to identify surfaces. We illustrate the method by examples.

2.1. **Application 1. Plane models.** Let $M$ be a surface whose plane model has a word $dac^{-1}bca^{-1}b^{-1}d^{-1}$. Plane model induces a cell decomposition on the surface $M$. By examining vertex identification, we have $\nu = 3, e = 4, f = 1$. Since the word is orientable and $\chi(M) = 3 - 4 + 1 = 0$, the surface $M$ must be a torus.

Let $M$ be a surface having a plane model with a word $ca^{-1}b^{-1}cdab^{-1}d$. This is a nonorientable word with $\chi(M) = 1 - 4 + 1 = -2$. Hence $M = 4P = N_4$.

2.2. **Application 2. Surfaces spanning knots and links.** See text Beginning Topology on pages 207 and 208 for pictures. To identify surfaces,

(1) Decide orientability of a given surface.
(2) Count the number of boundaries.
(3) Introduce a cell decvomposition of a surface and compute Euler characteristic. Here, the important point is that in this decomposition, each region must be homeomorphic to an open disc, that is, a 2-cell. If not, you have to cut regions further to make them into 2-cells.
(4) For a spanninbg surface $M$, let $\widetilde{M}$ be the closed surface obtained by capping boundaries. If there are $k$ boundaries, we have $\chi(\widetilde{M}) = \chi(M) + k$. Use this to identify $\widetilde{M}$ as $gT$ or $hP$.
(5) The original surface $M$ can be described as $\widetilde{M}$ with $k$ open discs removed.

Practice identifying spanning surfaces given on pages 207 and 208 above.

## 3. Regular Complexes on Surfaces

**Definition 3.1.** A regular complex on a surface $M$ is a cell decomposition on $M$ such that

(1) each face has the same number of edges $a \geq 3$,
(2) each vertex has the same valency $b \geq 3$,
(3) two faecs meet along a single edge, at a single vertex, or none at all,
(4) no two faces meet with itself.

A regular complex above on $M$ is denoted by $(a, b)\mathbb{M}$.

**Example 3.2.** Platonic solids are regular complexes on the sphere $S$.

(1) Tetrahedron $= (3, 3)\mathbb{S}$ with $\nu = 4, e = 6, f = 4$.
(2) Cube $= (4, 3)\mathbb{S}$ with $\nu = 8, e = 12, f = 6$.
(3) Octahedron $= (3, 4)\mathbb{S}$ with $\nu = 6, e = 12, f = 8$.
(4) Dedecahedron $= (5, 3)\mathbb{S}$ with $\nu = 20, e = 30, f = 12$.
(5) Icosahedron $= (3, 5)\mathbb{S}$ with $\nu = 12, e = 30, f = 20$.

**Lemma 3.3.** *Consider a regular complex of type $(a, b)$ on a surface $M$. Then*

$$af = 2e, \qquad \nu b = 2e.$$

*The Euler characteristic of $M$ is given by*

$$\chi(M) = 2e\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right).$$

**Proposition 3.4.** *Platonic solids are the only regular convex polytopes.*

*Proof.* Since $\chi(S^2) = 2$, we have $e(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}) = 1$. From this, only possibilities are $(a, b) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$. The numbers $\nu, e, f$ can be computed from $a, b$, and these are Platonic solids. □

**Definition 3.5.** Given a cell complex $C$ on a surface $M$, its dual cell complex $C'$ can be constructed as follows. Place a veretx in the middle of each face of $C$. There is an edge between two new vertices if and only if the corresponding faces of $C$ share a common edge. These new vertices and edges divide the surface $M$ into 2-cells which are the faces of $C'$.

Suppose cell complex is an $(a, b)\mathbb{M}$. In the dualization process, an $a$-gon face in $C$ becomes a vertex of valence $b$ in the dual complex $C'$. A vertex of valence $b$ in $C$ becomes a $b$-gon face. Thus, $C' = (b, a)\mathbb{M}$.

## 4. regular complexes on $\mathbb{P}, \mathbb{T}, \mathbb{K}$

**Proposition 4.1.** (1) *On the projective plane* $\mathbb{P}$*, all the possible regular complexes are of type* $(3, 5)$ *and* $(5, 3)$.

(2) *On the torus* $\mathbb{T}$*, all the possible regular complexes are of type* $(3, 6), (4, 4), (6, 3)$. *The number of edges, vertices and faces are not fixed, and there are infinitely many for each type.*

(3) *On the Klein bottle* $\mathbb{K}$*, all the possible regular complexes are of type* $(3, 6), (4, 4), (6, 3)$. *The number of edges, vertices and faces are not fixed, and there are infinitely many for each type.*

The projective plane can be obtained by identifying antipodal points on the sphere. Thus, roughly, we can think of the projective plane $P$ as "half" of the sphere $S^2$. The regular complex $(3, 5)\mathbb{P}$ and $(5, 3)\mathbb{P}$ are obtained as the half of the dedecahedron and icosahedron. The cell complex on the projective plane of type $(3, 3), (3, 4), (4, 3)$ are also obtained as the "half" of the tetrahadron, octahedron, and cube, although these cell complexes are not technically regular complexes.