

Notes on midterm, complex numbers and geometry.

1. A “root” of a polynomial is a number r such that $p(r) = 0$.

Example: $r = i$ and $r = -i$ are the roots of the polynomial $z^2 - 1$.

FUNDAMENTAL THEOREM OF ALGEBRA. A polynomial $p(z)$ of degree n with real or complex coefficients has n complex roots counted with multiplicity.

So $z^8 = 1$ has 8 roots ‘counted with multiplicity’? To say they are distinct, means they are all different.

What does “counted with multiplicity” mean and why would we count roots twice? The polynomial $z^2 = 0$ has only one solution, namely $z = 0$, so we must count it twice. $(z - 5)^1 0$.

The reason we count more than once has to do with polynomial division. If $z = r$ is a root, of the degree n polynomial, then the linear (equals ‘degree one’) polynomial $z - r$ must divide $p(z)$ (*). So $p(z) = (z - r)q(z)$ where q has degree $n - 1$. Now we look for a root of q . If we find one, say r_2 then $q(z) = (z - r_2)f(z)$. Continuing in this manner, we see that the fundamental theorem of algebra is equivalent to saying that we can write any polynomial $p(z)$ as

$$p(z) = A(z - r_1)(z - r_2) \dots (z - r_n)$$

where A, r_1, \dots, r_n are complex numbers. A ‘multiple root’ is then a root that appears twice or more in this list.

Because of this algebra fact, the FUNDAMENTAL THEOREM is often stated in the apparently more simple, but equivalent way: Every polynomial has a complex root.

Example: $p(x) = x^2 - 1$ has no real roots. It has two complex roots, $x = \pm i$.

Roots of unity. The n roots of unity are the n roots to the equation $z^n = 1$. They are situated on the unit circle $|z| = 1$ and are evenly spaced, starting at $z = 1$, and marching around the circle at jumps of angle $2\pi/n$. If we join their vertices we get a regular n -gon.

Why do the n -roots lie on the unit circle? For $z = x + iy$ write $|z| = \sqrt{x^2 + y^2}$. Then the unit circle is defined by the equation $|z| = 1$. Thus, we want to show that $z^n = 1$ implies $|z| = 1$. This follows from the fact that if z, w are two complex numbers then $|zw| = |z||w|$, a rather surprising fact, really. Applying this n times to the equation $|z^n| = 1$ yields $|z|^n = 1$. But $|z|$ is always a positive real number, and the only positive real number whose n th power is 1 is $|z| = 1$.

Why are they evenly spaced? I ask you to accept, for the moment, that multiplying by a unit modulus complex number is a rotation. Now let ω be any root, and let $ROOTS = \{r_1, \dots, r_n\}$ be the collection of all roots. Consider the map $z \mapsto \omega z$. It rotates the circle by some amount. It maps the collection $ROOTS$ of n th roots to itself: for if $r^n = 1$ is such a root, then $(r\omega)^n = r^n \omega^n = 1 \cdot 1 = 1$. Consequently multiplying the roots by any fixed root, rotates the circle, mapping the roots to themselves. By taking ω be what is called the ‘primitive’ n th root, the root lying in the first quadrant $x > 0, y > 0$ closest to the x -axis we find the spacing.

For the 6th roots of unity, see the text, p 214.

Some facts about roots