Mathematics 23A, Multivariable Calculus
Fall 2007, Richard Montgomery

Mock Final

The exam will have about 8 questions, several with multiple parts. One 8 and a 1/2 by 11 'cheat sheet" is allowed. No cell phones, calculators, books or more than this 1 page of notes is **not** allowed. Show your work. Justify claims when a proof is required.

1. **Problem 1** For each questions below, indicate if the statement is *true* (T) or *false* (F). You do **not** need to justify the answer.

   (a) If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ then there is a scalar field f such that $\mathbf{F} = \nabla f$.

   (b) If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable and everywhere negative, then it must have a critical point somewhere on $\mathbb{R}^3$.

   (c) There is an $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = x\mathbf{j}$.

   (d) If $f : \mathbb{R}^3 \to \mathbb{R}$ be a function which is twice continuously differentiable and if $\mathbf{F} = \nabla f$ Then $\nabla \cdot \mathbf{F} = \mathbf{0}$.

   (e) You can find three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ on the plane $\mathbb{R}^2$ all of unit length whose dot products with each other are zero.

   (f) $a\mathbf{i} + b\mathbf{j}$ and $b\mathbf{i} + a\mathbf{j}$ are perpindicular.

   (g) Let $f(x, y)$ be a smooth function which has a critical point at the origin. If the Hessian there is $-1/6$ times the identity matrix then the origin is a local maximum for f.

   (h) The flow lines of the vector field $F(x, y) = x\,\mathbf{i}$ are rays parallel to the y axis.

   (i) $\nabla \times (f\mathbf{F}) = \nabla f \times \mathbf{F} + f\ \nabla \times \mathbf{F}$, for a smooth function f on $\mathbb{R}^3$ and smooth vector field $\mathbf{F}$ on $\mathbb{R}^3$.

2. 2. (10 points) Find the equation of the tangent plane to the surface

$$\ln(xz) + x^2 e^{yz} + \cos(xy) = 2$$

   at the point $P = (1, 0, 1)$.

3. Consider the set of all lines in the plane parallel to the diagonal $x = y$. Among these lines some of them are tangent to the conic section $x^2 + 2xy + 2y^2 = 1$. Find all points of tangency.

   Extra credit (H.S.) Is this conic section an ellipse, hyperbola or parabola?

4. At points (x, y) along the curve $(x - 1)^2 y - x(y - 2)^2 = 0$ we can usually, but not always, solve (locally) for one variable in terms of the other one. For the points given below, decide whether it is possible to solve at all for one variable in terms of the other, and if so, for which variable in terms of which. Each question then has precisely one of four possible answers: 'none', "can solve for $x = x(y)$", " can solve for $y = y(x)$", or "can solve for both $x = x(y)$ and $y = y(x)$"

   a) $(x, y) = (0, 0)$.

   b) $(x, y) = (1, 2)$

   c) $(x, y) = (1, \frac{9}{4})$

5. You are given data for a pressure field $p(x, y, z)$ at and near the origin. The data are $p(0.00, 0.00, 0.00) = 0.00$, $p(0.10, 0.00, 0.00) = 0.15$, $p(0.10, 0.10, 0.00) = 0.40$, $p(0.10, 0.10, .010) = -0.25$. Use Taylor's theorem to estimate $p(0, 0.0, 0.1)$. HINT: assume the 1st order Taylor expansion is accurate, and use it to derive what the gradients at the origin are.

6. (10 points) Consider the curve $x = t, y = 2t^2, z = t^3$, $0 \leq t \leq 1$.

   a) What are the endpoints of this curve?

   b) Set up the integral for computing the arclength of this curve. DO NOT ATTEMPT TO EVALUATE.

   c) Show that the integral in b is greater than $\sqrt{6}$.

7. Find all values of the parameter k for which the function $f(x, y) = x^2 + kxy + 2y^2$ has the origin as a saddle point.

   Find all values for which the origin is a local maximum.

8. Let $f(x, y) = x^2 + y^2$ Find the constrained extrema of f on the ellipsoid $x^2 - xy + 4y^2 + z^2 = 1$. Use the method of Lagrange multipliers.

9. Find the absolute maximum and minimum values of $f(x, y) = 4x^2 - xy + 4y^2$ on the disk $D = \{(x, y)|x^2 + y^2 \leq 2\}$.

10. (30 points) On the "next page" [go to: http://math.ucsc.edu/ rmont/classes/23A/vflds1.pdf] you will find a picture of six vector fields, labelled (a) through (f). Here you will find formulae for more thirteen vector fields, (1) - (13). Match the picture with the formula. For each letter (a) through (f), choose the number that corresponds to it.

    (1) $\mathbf{i}$

    (2) $x\mathbf{i}$

    (3) $x^2\mathbf{i}$

    (4) $-x^2\mathbf{i}$

    (5) $\sin(x)\mathbf{i}$

    (6) $\sin(y)\mathbf{i}$

    (7) $\sin(y)\mathbf{j}$

    (8) $\sin(x)\mathbf{j}$

    (9) $x\mathbf{i} + y\mathbf{j}$

    (10) $x\mathbf{i} - y\mathbf{j}$

    (11) $y\mathbf{i} + x\mathbf{j}$

    (12) $y\mathbf{i} - x\mathbf{j}$

    (13) $-y\mathbf{i} + x\mathbf{j}$

    (14) $\frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j}$