Problem:

Find a unit speed parameterization of the circle defined by intersecting the unit sphere with the plane $x + y + z = 1$

Solution. We are given a hint: the intersection is a circle.

Start by thinking about the the unit circle in the xy plane. It is parameterized as $\gamma(t) = \cos t\mathbf{e_1} + \sin(t)\mathbf{e_2}$ where $\mathbf{e_1}, \mathbf{e_2}$ are the standard $\mathbf{i}, \mathbf{j}$, or indeed ANY orthonormal basis for the xy plane. Routine generalization and a bit of thought now show us how to parameterize any circle, anywhere in space. We will need to know its center, $\mathbf{c}$, its radius, $r$ and the plane $P$ on which it sits. Since $P$ must contain the center $\mathbf{c}$ we can view $P$ as the translate of a plane $V$ through the origin ( a two-dimensional linear subspace) by $\mathbf{c}$: thus: $P = \mathbf{c} + V$. To get the parameterization we will need to find an orthonormal [o.n.] basis $\mathbf{e_1}, \mathbf{e_2}$ for $V$. The given circle is then parameterized by

$$\gamma(t) = \mathbf{c} + r(\cos t\mathbf{e_1} + \sin(t)\mathbf{e_2}) \quad (**)$$

Our case: we are given $P$ which is $x + y + z = 1$. Clearly $V$ is given by $x + y + z = 0$. We must find $\mathbf{c}$ and $r$. The normal vector to $P$ is $(1, 1, 1)$. By symmetry, or by geometry, the center is $\mathbf{c} = \lambda(1, 1, 1)$ where $\lambda$ is the unique scale factor which places the point $\mathbf{c}$ on the plane $P$. Thus $3\lambda = 1$ and $\lambda = (1/3)$. The distance $d$ of $\mathbf{c}$ from the origin is $d = |\mathbf{c}| = \sqrt{3\lambda^2} = 1/\sqrt{3}$. To find the radius of the circle $r$, consider a random point $\mathbf{q}$ on the circle. The points $\mathbf{0}, \mathbf{c}, \mathbf{q}$ form a right triangle, with the line segment from $\mathbf{0}$ to $\mathbf{q}$ being the hypotenuse. But $|\mathbf{q}| = 1$ since we know that $\mathbf{q}$ lies on the unit sphere $x^2 + y^2 + z^2 = 1$. Upon applying Pythagoras to the right triangle we see that $1^2 = d^2 + r^2$ so that $r = \sqrt{2/3}$.

It remains to find an o.n. basis $\mathbf{e_1}, \mathbf{e_2}$ for the linear subspace $V$ defined by $x + y + z = 0$. Choose any vector on the plane, say $\mathbf{v} = (1, -1, 0)$. To find a perpindicular to $\mathbf{v}$ lying on the plane, make the guess $\mathbf{w} = (1, 1, C)$ and plug in to the equation of the plane to get $C = -2$. Then $\mathbf{v}, \mathbf{w}$ are orthogonal (= perpindicular) vectors lying on the plane $V$. To make them orthonormal (perpindicular and unit length) we need only normalize them:

$$\mathbf{e_1} = (1/\sqrt{2})(1, -1, 0)$$

$$\mathbf{e_2} = (1/\sqrt{6})(1, 1, -2)$$

Plugging in the values we have found for $\mathbf{c}$, $r$ and $\mathbf{e_1}, \mathbf{e_2}$ into (**) will give a parameterization. But it will not be unit speed, indeed: $|d\gamma(t)/dt| = r$. So to make it unit speed, replace $t$ occurring in the trig functions by $t/r$. Thus a unit speed parameterization of the circle is

$$\gamma(t) = (1/3, 1/3, 1/3) + \sqrt{2/3}[(1/\sqrt{2})(1, -1, 0)\cos(\sqrt{3/2}t) + (1/\sqrt{6})(1, 1, -2)\sin(\sqrt{3/2}t)].$$

(Other solutions are possible, simply by finding other o.n. basis $\mathbf{e_1}, \mathbf{e_2}$ for $V$. )